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THE EFFECTIVENESS OF DAMPERS FOR THE ANALYSIS OF EXTERIOR SCALAR WAVE DIFFRACTION BY CYLINDERS AND ELLIPSOIDS

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SUMMARY

The technique of dampers is now widely used in the numerical analysis of unbounded problems. The dampers are used to absorb the outgoing waves. This paper will consider in detail the best formulations of damper methods. Examples will be given showing the effectiveness of three different dampers, in two dimensional and three dimensional models. The geometries considered are circular and elliptical cylinders, spheres and ellipsoids. The results indicate that dampers are indeed very effective, particularly those of higher order, which have recently been developed by Bayliss *et al.*[1,2] and others.

1. INTRODUCTION

The types of wave problem for which these dampers can be directly used are exterior scalar wave problems. Important examples of these include

(i) surface water waves—wave forces on offshore structures, diffraction and refraction of waves in the coastal zone
(ii) pressure waves in fluids—depth charge problems, sonar, fluid–structure interaction
(iii) pressure waves in elastic media—earthquakes and vibration
(iv) electromagnetic waves—aerials, waveguides.

The two examples considered in the remainder of the paper are linear free surface waves in two dimensions and compression waves in three dimensions. Such waves are governed by the Helmholtz equation

$$\nabla^2\phi + k^2\phi = 0 \tag{1}$$

where ϕ is the scalar wave variable, and k is the wave number, given by the frequency, ω, divided by the wave speed, c. In water of constant depth, h, with an acceleration due to gravity, g, the velocity of the surface wave, c is $(gh)^{1/2}$. The velocity of compressive waves is $c = (K/\rho)^{1/2}$, where K is the bulk modulus and ρ is the density. The technique of dampers is of course applicable to vector wave propagation, but this will not be considered here.

2. EXTERIOR WAVE PROBLEM

Waves are considered whose wavelength is long compared with the fluid depth and whose amplitude is small. Assuming that the fluid is incompressible and motion is irrotational, the

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wave equation governing the flow can be derived from the continuity equation and the momentum equations in the form

$$\frac{\partial}{\partial x}\left[h\frac{\partial \phi}{\partial x}\right]+\frac{\partial}{\partial y}\left[h\frac{\partial \phi}{\partial y}\right]-\frac{1}{g}\frac{\partial^2 \phi}{\partial t^2}=0 \tag{1}$$

A more general theory which includes long waves and short waves has been developed by Berkhoff[3] and has been implemented in the finite element context by Bettess and Zienkiewicz.[4] Berkhoff's wave expression corresponding to equation (1) is

$$\frac{\partial}{\partial x}\left[cc_g\frac{\partial \phi}{\partial x}\right]+\frac{\partial}{\partial y}\left[cc_g\frac{\partial \phi}{\partial y}\right]+\frac{\omega^2 c_g}{c}\phi=0 \tag{2}$$

where c is the wave speed and c_g is the group velocity. As the theme of this paper is to compare the accuracy of these dampers, the simple constant depth model expressed by equation (1) is adopted. In the case of constant depth equation (1) reduces to

$$\frac{\partial^2 \bar{\phi}}{\partial x^2}+\frac{\partial^2 \bar{\phi}}{\partial y^2}+k^2\bar{\phi}=0 \tag{3}$$

where $k=\omega/(gh)^{1/2}$ is wave number and ω is angular frequency. In the above, it is assumed that velocity potential is periodic, e.g.

$$\phi(x, y, z, t)=\bar{\phi}(x, y, z)e^{-i\omega t} \tag{4}$$

Thus wave motion is generally described by the Helmholtz equation. When the scattering of surface waves due to circular or elliptical cylinders is calculated, equation (3) will have to be solved.

On the other hand, in three dimensional problems the scattering of waves, such as sound waves or compressive waves, diffracted by a sphere or ellipsoid in homogeneous infinite domains is dealt with. Then the basic equation to be solved is

$$\frac{\partial^2 \bar{\phi}}{\partial x^2}+\frac{\partial^2 \bar{\phi}}{\partial y^2}+\frac{\partial^2 \bar{\phi}}{\partial z^2}+k^2\bar{\phi}=0 \tag{5}$$

The boundary condition at infinity will be dealt with in the following section.

3. RADIATION BOUNDARY CONDITION

The boundary condition at infinity is required in order to solve potential problems described by the Helmholtz equation in an infinite domain. This condition is the so-called Sommerfeld's radiation condition.[5] The uniqueness of the solution satisfying this condition has been proved for periodic problems.[6]

Sommerfeld's radiation condition corresponding to equations (3) or (5) can be expressed as

$$\lim_{r\to\infty} r^{(h-1)/2}\left[\frac{\partial \bar{\phi}}{\partial r}-ik\bar{\phi}\right]=0 \tag{6}$$

or

$$r^{(h-1)/2}\left[\frac{\partial \bar{\phi}}{\partial r}-ik\bar{\phi}\right]=O(1/r), \qquad r\to\infty \tag{7}$$

where r is the distance from any fixed point and h is the number of dimensions, and $O(f(r))$ means a quantity of order not greater than $f(r)$ for large r. However it is difficult to incorporate equations (6) or (7) directly into FEM programs, since they are in the form of a limit. Here we describe three forms of the radiation condition corresponding to the number of dimensions involved. This explanation should clarify the property of each damper.

3.1. *One dimensional problems*

The fundamental solution for the one dimension Helmholtz equation is $\exp(ikr)$ where r is the distance from a singular point. Now the potential ϕ can be in general written as

$$\phi = F_1(r-ct)+G_1(r+ct) \tag{8}$$

In the equation above F_1 stands for outgoing waves and G_1 for incoming ones. As only the outgoing waves should satisfy the radiation condition, G_1 is now required to vanish. By eliminating F_1, we obtain

$$\frac{\partial\bar{\phi}}{\partial r}+\frac{1}{c}\frac{\partial\bar{\phi}}{\partial t}=0 \tag{9}$$

For periodic motion this condition becomes

$$\frac{\partial\bar{\phi}}{\partial r}-ik\bar{\phi}=0 \tag{10}$$

This boundary condition can be easily incorporated in FEM programs as a plane damper, as was shown by Zienkiewicz and Newton.[7]

3.2. *Two dimensional problems*

For two dimensional problems, the derivation is more complicated than for one dimensional problems. This is because the two dimensional fundamental solution is the Hankel function and the potential ϕ cannot be written strictly in such a form as equation (8). But the Hankel function of the first kind of order zero. $H_0^1(kr)$ can be approximately expressed for large r as follows:

$$H_0^1(kr)\sim \exp(i(kr-\pi/4))\left(\frac{2}{\pi kr}\right)^{1/2}=C\left(\frac{1}{r}\right)^{1/2}e^{ikr} \tag{11}$$

where C is a constant independent of r. When the periodic term $\exp(-i\omega t)$ is taken into account, for large r the potential ϕ can be written as

$$\phi = F_2(r-ct)/(r)^{1/2} \tag{12}$$

In the same way as for the one dimensional problems, the following simple radiation conditions can be obtained.

$$\frac{\partial\phi}{\partial r}+\frac{1}{2r}\phi+\frac{1}{c}\frac{\partial\phi}{\partial t}=0 \tag{13}$$

or

$$\frac{\partial\bar{\phi}}{\partial r}+\frac{1}{2r}\bar{\phi}-ik\bar{\phi}=0 \tag{14}$$

Equations (13) and (14) describe the radiation conditions for the outgoing waves at infinity which progress uniformly in any direction. We will call the damper formulated by equation (14) the cylindrical damper.

3.3. *Three dimensional problems*

In the same way as in the one dimensional problem the potential ϕ can be generally written as

$$\phi = F_3(r-ct)/r \tag{15}$$

and the radiation condition for three dimensional problems is

$$\frac{\partial\phi}{\partial r}+\frac{1}{r}\phi+\frac{1}{c}\frac{\partial\phi}{\partial t}=0 \tag{16}$$

or

$$\frac{\partial\bar{\phi}}{\partial r}+\frac{1}{r}\bar{\phi}-\mathrm{i}k\bar{\phi}=0 \tag{17}$$

Equation (17) is derived under the assumption that outgoing waves propagate uniformly in all directions. The damper described by equation (17) is here called a spherical damper.

These boundary conditions described above should be imposed at infinity. However these dampers are placed not at infinity but at a finite distance. One of the main purposes is therefore the comparison of the accuracy when each damper element is set up at a finite distance from distrubance such as a cylinder.

4. HIGH ORDER DAMPER THEORY

Recently a theory has been developed by Bayliss. Gunzberger and Turkel for higher order damper boundary conditions.[1] For completeness the entire necessary theory will be outlined here. The first essential step is to prove that any wave can be expressed in series form, following Atkinson[6] and Wilcox.[8] Next the series form is used to obtain a series of operators which then define a set of dampers of increasing order.[1,2] The proofs are different for two and three dimensions. As the three dimensional proof is essentially simpler and more natural it will be stated first.

Three dimensional damper theory

The starting point is Green's second identity.

$$\int_\Omega (u\nabla^2 v - v\nabla^2 u)\,\mathrm{d}\Omega = \int_S (u\nabla v - v\nabla u)\,\mathrm{d}s \tag{18}$$

where u and v are any two functions, which are defined in the unbounded domain, Ω, exterior to the surface S, and which satisfy the following conditions as the radius, r, tends to infinity

$$ru,\quad r^2\frac{\partial u}{\partial x},\quad r^2\frac{\partial u}{\partial y},\quad r^2\frac{\partial u}{\partial z},\quad rv,\quad r^2\frac{\partial v}{\partial x},\quad r^2\frac{\partial v}{\partial y},\quad r^2\frac{\partial v}{\partial z}$$

are bounded in absolute value for sufficiently large radius, r.

Let $u=\exp(ikr)/r$, the Green's function for the Helmholtz equation in three dimensions, and consider the geometry shown in Figure 1. The domain Ω is unbounded, and the boundary S consists of two parts, S_1, the surface of a sphere of radius $r=a$, and S_2, the

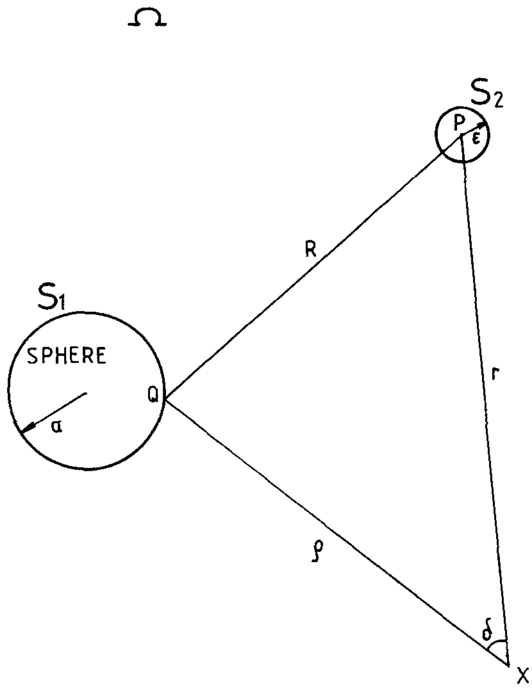

Figure 1. Geometry

surface of a sphere of radius ε, centred about the pole, P. v is taken to be the solution of the Helmholtz equation throughout Ω. Now the volume integral of equation (18) reduces to zero, and so

$$\int_{S_1+S_2}\left[\frac{e^{ikR}}{R}\frac{\partial v}{\partial n}+v\frac{\partial}{\partial r}\left[\frac{e^{ikR}}{R}\right]\right]\mathrm{d}s=0 \tag{19}$$

where R is the distance from the pole to a point Q on the surface S_1.

Now let the sphere of radius ε be shrunk to a point. Clearly $\mathrm{d}s=4\pi\varepsilon^2$, and the S_2 term becomes

$$\left[\frac{e^{ik\varepsilon}}{\varepsilon}\frac{\partial v}{\partial n}+v\left(ik-\frac{1}{\varepsilon}\right)\frac{e^{ik\varepsilon}}{\varepsilon}\right]4\pi\varepsilon^2=-4\pi v \tag{20}$$

So now

$$v=\frac{1}{4\pi}\int_{S_1}\frac{e^{ikR}}{R}\left[\frac{\partial v}{\partial n}+v\left(ik-\frac{1}{R}\right)\right]\mathrm{d}s \tag{21}$$

We will expand v about a fixed point X. Let r, ρ, δ denote PX, XQ and $\angle$PXQ, respectively. Then

$$\begin{aligned}
x&=r^{-1}\\
R&=(r^2-2r\rho\cos\delta+\rho^2)^{1/2}\\
R-r&=r\left[\left[1-\frac{2\rho\cos\delta}{r}+\frac{\rho^2}{r^2}\right]^{1/2}-1\right]\\
&=\frac{1}{x}\{(1-2x\rho\cos\delta+x^2\rho^2)^{1/2}-1\}\\
r/R&=(1-2x\rho\cos\delta+x^2\rho^2)^{-1/2}\\
ik-1/R&=ik-x(1-2x\rho\cos\delta+x^2\rho^2)^{-1/2}
\end{aligned} \tag{22}$$

Hence

$$\frac{e^{ikR}}{R}\Big/\frac{e^{ikr}}{r}=(1-2x\rho\cos\delta+x^2\rho^2)^{-1/2}\exp\frac{ik}{x}\{(1-2x\rho\cos\delta+x^2\rho^2)^{1/2}-1\} \tag{23}$$

$\frac{e^{ikR}}{R}\Big/\frac{e^{ikr}}{r}$ may be expanded in a power series because the right-hand side is an analytic function of x in the region $|2x\rho\cos\delta-x^2\rho^2|<1$. Assuming that v has continuous second derivatives on $S_1\cdot\left[\frac{e^{ikR}}{R}\right]\left[\frac{e^{ikr}}{r}\right]^{-1}\frac{\partial v}{\partial n}$ is also analytic. It follows that the following integral may be expanded in a power series.

$$\left[\frac{e^{ikr}}{r}\right]^{-1}\int_{S_1}\left[\frac{e^{ikR}}{R}\right]\frac{\partial v}{\partial n}\,ds=\sum_{n=0}^{\infty}c'_n x^n=\sum_{n=0}^{\infty}c'_n r^{-n} \tag{24}$$

The same result holds for $\left[\frac{e^{ikr}}{r}\right]^{-1}\int_{S_1}\frac{e^{ikR}}{R}\left(ik-\frac{1}{R}\right)v\,ds$. So

$$\left[\frac{e^{ikr}}{r}\right]^{-1}v=\sum_{n=0}^{\infty}c_n r^{-n} \tag{25}$$

Therefore v can be expanded in a series form as follows:

$$v=e^{ikr}\sum_{n=1}^{\infty}d_n r^{-n} \tag{26}$$

It is proved that the solution to the three dimensional Helmholtz equation can be written in the expansion form.

Next we will obtain a sequence of boundary conditions which annihilate the first m terms in the asymptotic expansion. The solution ϕ to the three dimensional Helmholtz equation can be expressed as

$$\phi=e^{ikr}\sum_{j=1}^{\infty}\frac{f_j(\theta,\phi)}{r^j} \tag{27}$$

The operator L

$$L=-ik+\frac{\partial}{\partial r} \tag{28}$$

is defined. When equation (27) is multiplied by r^m, equation (29) is obtained.

$$r^m\phi=e^{ikr}\sum_{j=1}^{m}r^{m-j}f_j(\theta,\phi)+e^{ikr}\sum_{j=m+1}^{\infty}r^{m-j}f_j(\theta,\phi) \tag{29}$$

Applying the operator L^m to both sides of equation (29), the first sum of the right-hand side of equation (29) obviously becomes zero. Hence

$$L^m(r^m\phi)=O(r^{-m-1}) \tag{30}$$

Equation (30) shows that the operator L^m can annihilate the first m terms in $r^m\phi$.

The operator acting only on ϕ should be obtained. Now we separate the operator L into the constant part L_1 and the differential part L_2 as follows

$$L_1 = -ik$$
$$L_2 = \frac{\partial}{\partial r} \tag{31}$$

and write $L^m(r^m\phi)$ explicitly when $m=1$ and $m=2$.

$m=1$

$$\begin{aligned} L(r\phi) &= (L_1+L_2)(r\phi) \\ &= L_1 r\phi + rL_2\phi + \phi L_2 r \\ &= r(L_1+L_2)\phi + \phi \\ &= rL\phi + \phi \end{aligned} \tag{32}$$

or

$$L(r\phi)/r = (L+1/r)\phi \tag{33}$$

$m=2$

$$\begin{aligned} L^2(r^2\phi) &= (L_1+L_2)^2(r^2\phi) \\ &= (L_1+L_2)\{L_1 r^2\phi + r^2 L_2\phi + 2r\phi\} \\ &= r^2(L_1+L_2)^2\phi + 4r(L_1+L_2)\phi + 2\phi \\ &= r^2L^2\phi + 4rL\phi + 2\phi \end{aligned} \tag{34}$$

or

$$\begin{aligned} L^2(r^2\phi)/r^2 &= L^2\phi + 4L\phi/r + 2\phi/r^2 \\ &= (L+3/r)(L+1/r)\phi \end{aligned} \tag{35}$$

On the analogy of equations (33) and (35) the new operator B_m is defined recursively as follows.

$$\begin{aligned} B_m &= \prod_{j=1}^{m}\left[L + \frac{2j-1}{r}\right] \\ &= \prod_{j=1}^{m}\left[-ik + \frac{\partial}{\partial r} + \frac{2j-1}{r}\right] \end{aligned} \tag{36}$$

It is easy to prove that B_m defined above annihilates the first m terms in the expansion (26). It then follows that

$$B_m\phi = O(r^{-2m-1}) \tag{37}$$

Thus the boundary condition

$$B_m\phi = 0 \tag{38}$$

matches the solution to the first m terms in equation (27) and it is clear that the more accurate results can be obtained by applying the higher order boundary conditions.

One difficulty is that the infinite series of operators in equation (36) generates higher and higher derivatives with respect to r. This leads to difficulties in a finite element model. However the order of the highest derivative can be reduced by using the Helmholtz equation itself. Now the following boundary condition is considered:

$$B_2\phi = 0 \tag{39}$$

The B_2 operator is

$$B_2=\left(-ik+\frac{\partial}{\partial r}+\frac{3}{r}\right)\left(-ik+\frac{\partial}{\partial r}+\frac{1}{r}\right)$$
$$=\left(\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r}+\frac{2}{r^2}-k^2-2ik\frac{\partial}{\partial r}-\frac{4ik}{r}\right) \tag{40}$$

Helmholtz's equation may be written in spherical co-ordinates (r, θ, δ) as

$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\frac{\partial\phi}{\partial r}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial\phi}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\delta^2}+k^2\phi=0 \tag{41}$$

$\frac{\partial^2\phi}{\partial r^2}$ can be eliminated from equations (40) and (41), so that

$$B_2\phi=\frac{2}{r}\frac{\partial\phi}{\partial r}-2ik\frac{\partial\phi}{\partial r}+\left(\frac{2}{r^2}-2k^2-\frac{4ik}{r}\right)\phi$$
$$-\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}-\frac{\cos\theta}{r^2\sin\theta}\frac{\partial\phi}{\partial\theta}-\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\delta^2}=0 \tag{42}$$

For simplicity only axisymmetric problems will be dealt with, and the potential ϕ is assumed to be independent of the δ co-ordinate. Then equation (42) is reduced to

$$B_2\phi=\frac{2}{r}\frac{\partial\phi}{\partial r}-2ik\frac{\partial\phi}{\partial r}+\left(\frac{2}{r^2}-2k^2-\frac{4ik}{r}\right)\phi-\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}-\frac{\cos\theta}{r^2\sin\theta}\frac{\partial\phi}{\partial\theta}=0 \tag{43}$$

Further, the artificial surface on which the boundary condition (39) is imposed is assumed to be spherical and the distance along the boundary s is introduced:

$$s=r\theta \tag{44}$$

Equation (43) can be rewritten as

$$\left(\frac{2}{r}-2ik\right)\frac{\partial\phi}{\partial r}+\left(\frac{2}{r^2}-2k^2-\frac{4ik}{r}\right)\phi-\frac{\partial^2\phi}{\partial s^2}-\frac{\cos\theta}{r\sin\theta}\frac{\partial\phi}{\partial s}=0 \tag{45}$$

or

$$\frac{\partial\phi}{\partial r}+\alpha\phi-\beta\frac{\partial^2\phi}{\partial s^2}-\beta\frac{\cos\theta}{r\sin\theta}\frac{\partial\phi}{\partial s}=0 \tag{46}$$

where

$$\alpha=\left(\frac{2}{r^2}-2k^2-\frac{4ik}{r}\right)\Big/\left(\frac{2}{r}-2ik\right)$$
$$=\frac{1}{r}-ik$$
$$\beta=\frac{1}{2\alpha}$$

Now for the element formulation, the boundary integral

$$A=\int_0^{2\pi}\!\!\int\left[\frac{\alpha}{2}\phi^2+\frac{\beta}{2}\left(\frac{\partial\phi}{\partial s}\right)^2\right]d\Gamma \tag{47}$$

is used. Considering that ϕ is independent of δ, equation (47) is reduced to

$$\begin{aligned} A &= \int\int_0^{2\pi} \left[\frac{\alpha}{2}\phi^2 + \frac{\beta}{2}\left(\frac{\partial\phi}{\partial s}\right)^2\right] r \sin\theta \, d\delta \, ds \\ &= \int \left[\frac{\alpha}{2}\phi^2 + \frac{\beta}{2}\left(\frac{\partial\phi}{\partial s}\right)^2\right] 2\pi r \sin\theta \, ds \\ &= \int F \, ds \end{aligned} \tag{48}$$

The variation of F generates the boundary condition.[9]

$$\begin{aligned} [F]_\phi &= \frac{\partial F}{\partial\phi} - \frac{\partial}{\partial s}\left[\frac{\partial F}{\partial(\partial F/\partial s)}\right] \\ &= \left(\alpha\phi - \beta\frac{\partial^2\phi}{\partial s^2} - \beta\frac{\cos\theta}{r\sin\theta}\frac{\partial\phi}{\partial s}\right) 2\pi r \sin\theta \\ &= 0 \end{aligned} \tag{49}$$

The term $\frac{\partial\phi}{\partial r}$ arises as the natural boundary condition.

Equation (48) can be easily incorporated into FEM programs as higher order damper elements. Note that the first order operator B_1 is identical to the spherical damper defined in equation (17).

Two dimensional damper

For two dimensions the convergent expansion is given by Karp[10] as follows:

$$\phi = H_0(kr)\sum_{j=0}^{\infty}\frac{F_j(\theta)}{r^j} + H_1(kr)\sum_{j=0}^{\infty}\frac{G_j(\theta)}{r^j} \tag{50}$$

where H_0 and H_1 are the Hankel functions of the first kind of orders 0 and 1. However these are not easy to work with. So another series, which is asymptotically true for large r, is adopted.

$$\phi = \left(\frac{2}{\pi kr}\right)^{1/2} e^{i(kr-\pi/4)} \sum_{j=0}^{\infty}\frac{f_j(\theta)}{r^j} \tag{51}$$

This equation corresponds to equation (27) for three dimensions. This expansion leads to a series of operators in the same way as in three dimensions.

$$B_m = \prod_{j=1}^{m}\left[\frac{\partial}{\partial r} + \frac{(2j-3/2)}{r} - ik\right] \tag{52}$$

$$B_m\phi = O(r^{-2m-\frac{1}{2}}) \tag{53}$$

Now the boundary condition specified by operator B_2 is considered:

$$B_2\phi = 0 \tag{54}$$

If the boundary is strictly circular, equation (54) can be expressed as follows:

$$\frac{\partial\phi}{\partial r} + \alpha\phi - \beta\frac{\partial^2\phi}{\partial s^2} = 0 \tag{55}$$

where

$$\alpha = \left[\frac{3}{4r^2} - 2k^2 + \frac{3ik}{r}\right] \Big/ \left[\frac{2}{r} + 2ik\right]$$

$$\beta = 1 \Big/ \left[\frac{2}{r} + 2ik\right]$$

Therefore for an element implementation, the line integral

$$A = \int \left[\frac{\alpha}{2}\phi^2 + \frac{\beta}{2}\left(\frac{\partial \phi}{\partial s}\right)^2\right] \mathrm{d}s \tag{56}$$

is obtained.[9,11] The two dimensional higher order damper discussed in this paper is based on equation (56). It is important that the first order operator B_1 coincides with the cylindrical damper described by equation (11).

5. NUMERICAL RESULTS

It is interesting to estimate the errors of these dampers in different parameter ranges.

1. The error for fixed k and m as the position of the artificial surface, i.e. r_1, varies.
2. The error for fixed k and r_1 as the order of the boundary operator m increases.
3. The error for fixed r_1 and m as the wave number, k increases.

In view of this, four problems were solved to check the accuracy and the behaviour of each damper.

First the surface waves scattered by a cylinder standing in shallow water with constant depth were analysed and the three items above were investigated (circular cylinder problem). Further, in order to compare the effectiveness of each damper the following three problems were solved: surface waves hitting an elliptical cylinder (elliptical cylinder problem), and waves scattered by a sphere or an ellipsoid in a three dimensional homogeneous domain (sphere and ellipsoid problems, respectively). The radii of the cylinders and spheres considered here are 1·0 and the major and the minor axis lengths of the elliptical cylinder and the ellipsoid are 2·0 and 1·0, respectively. As the basic solution, the analytical solution[12] is used in the cylinder and sphere problems, the numerical solution by using fine finite elements and boundary integrals is adopted for the elliptical cylinder problem and the numerical solution by using fine finite elements and infinite elements for the ellipsoid problem. All three dimensional examples were analysed as axisymmetric problems.

5.1. *Circular cylinder problem*

Figures 2–4 show the numerical results when the wave number k varies. The real and imaginary parts of the wave elevation around a cylinder are illustrated. The relative error around a cylinder, which is defined by $(|\eta_n| - |\eta_a|)/|\eta_a|$ where η_n, η_a are the numerical and analytical values respectively, are shown in Table I. The finite element mesh used is shown in Figure 5(i). It is seen that the results by using higher order dampers are quite close to theoretical values in any case and the errors of plane dampers are larger than the others.

Next, examples were calculated in which the wave number is considered to be 1·0 and the outer radius and the number of elements in the radial direction are changed. The meshes used when the outer radii are 1·5, 4·0 and 7·0 are shown in Figure 5. Table II and Figure 6 show the relative errors around a cylinder produced using each damper. It is seen that the

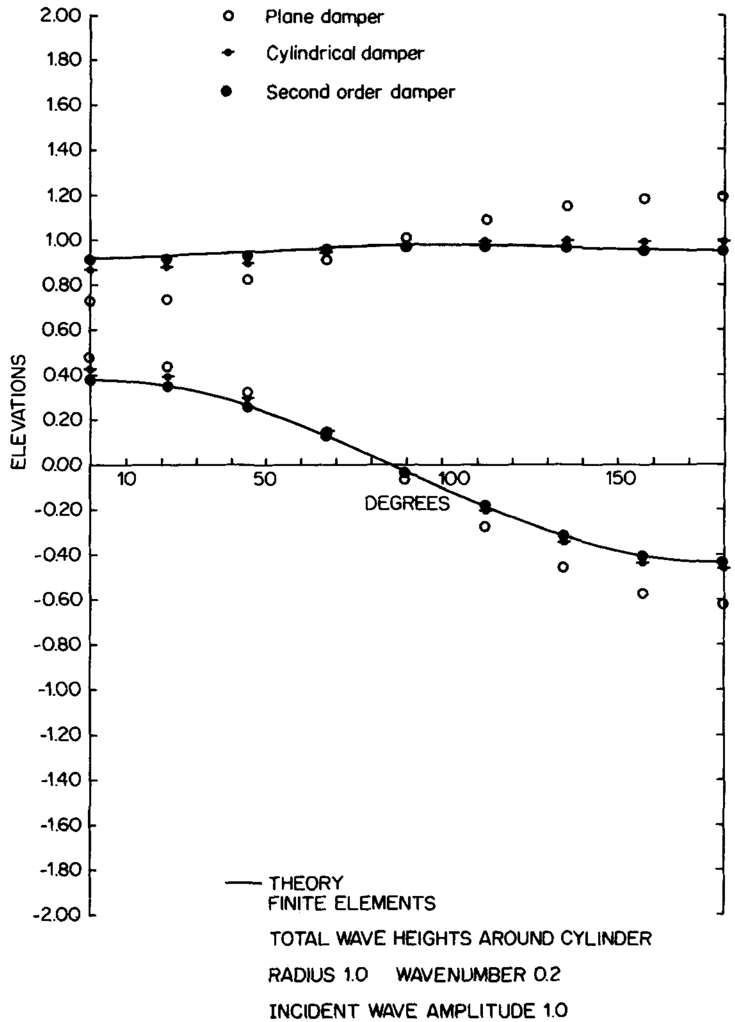

Figure 2. Comparison of wave diffraction by a cylinder using three types of dampers ($k = 0{\cdot}2$, $r_{out} = 1{\cdot}5$)

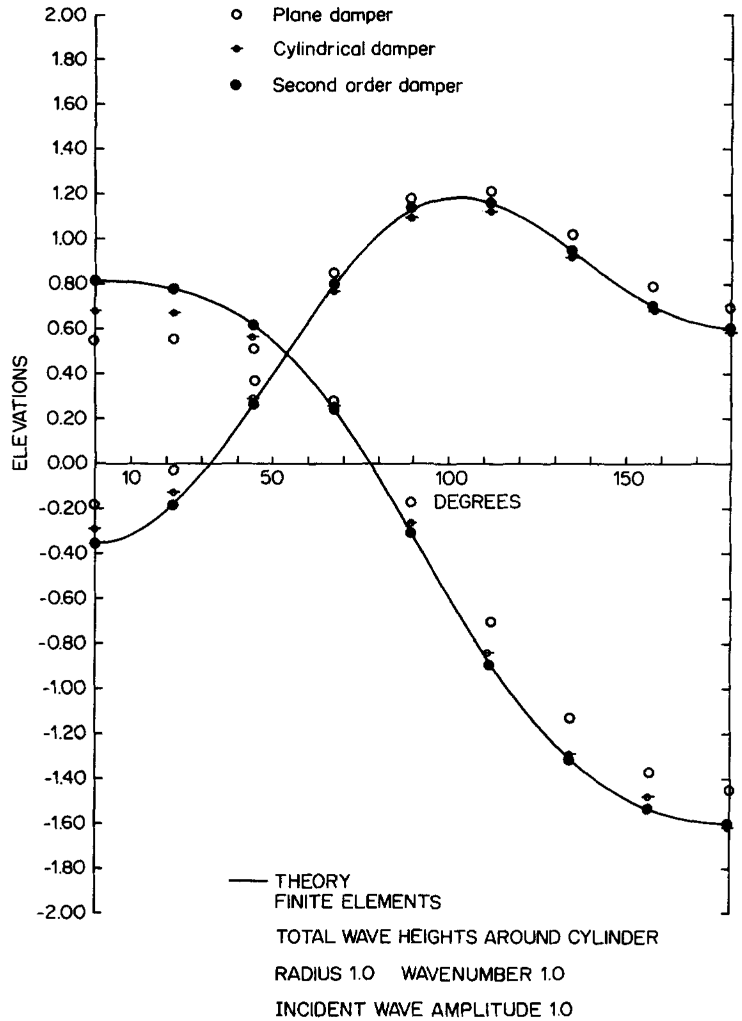

Figure 3. Comparison of wave diffraction by a cylinder using three types of dampers ($k = 1{\cdot}0$, $r_{out} = 1{\cdot}5$)

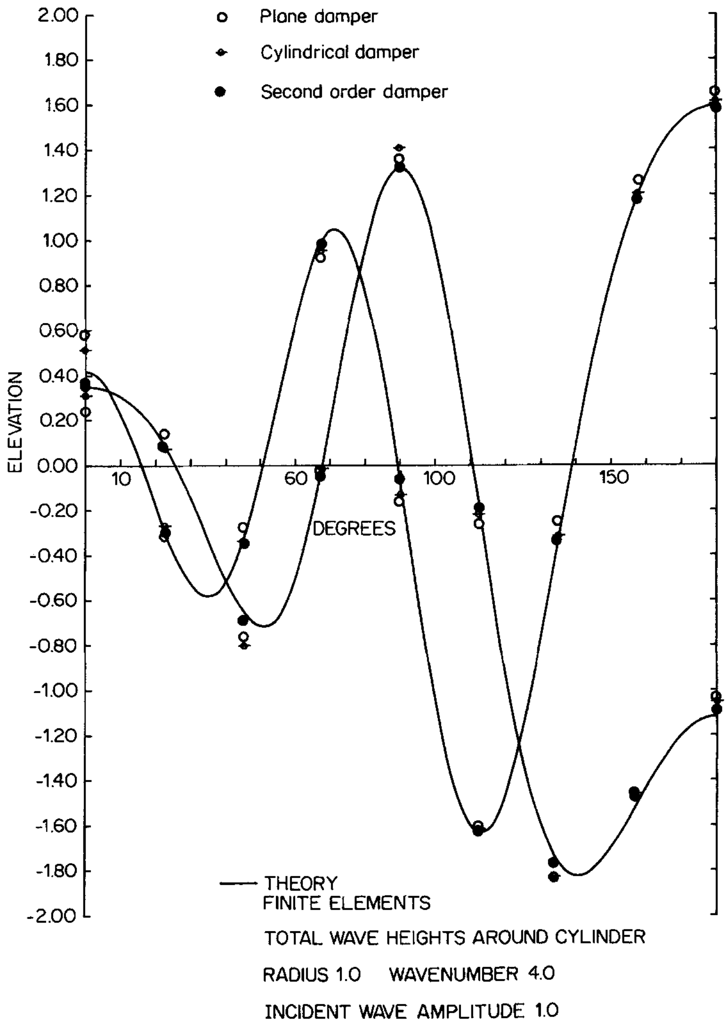

Figure 4. Comparison of wave diffraction by a cylinder using three types of dampers ($k = 4{\cdot}0$, $r_{out} = 1{\cdot}5$)

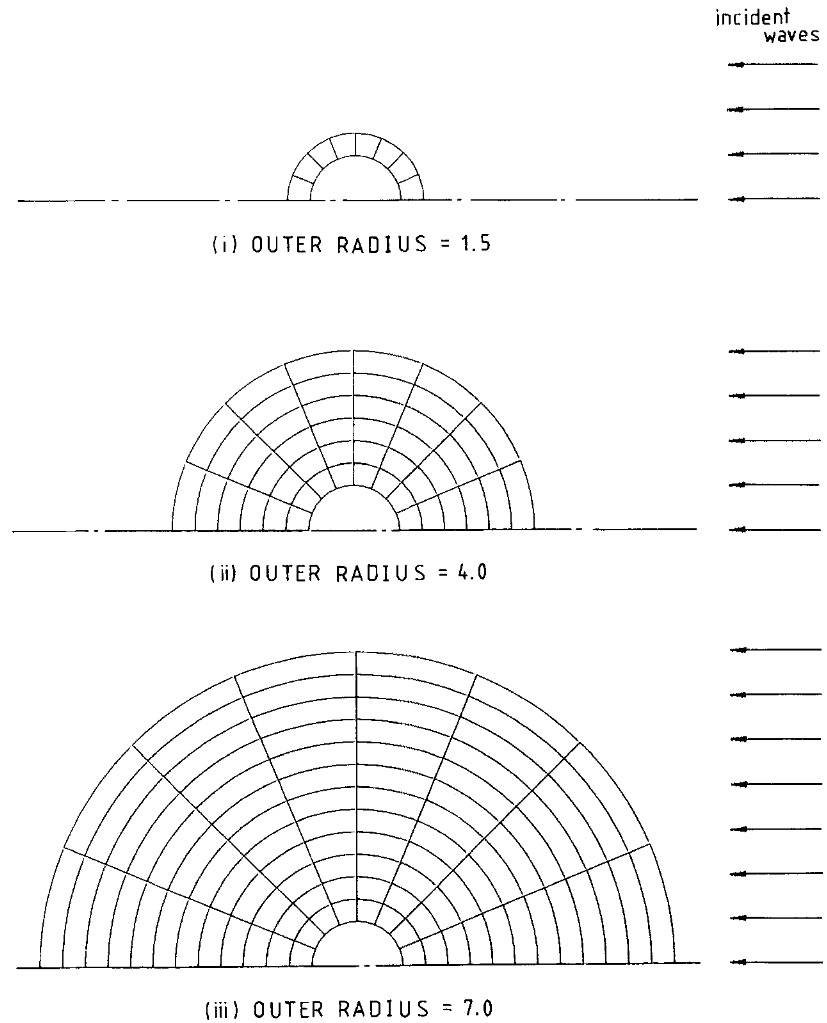

Figure 5. Element meshes

Table I. Relative (percentage) errors in circular cylinder problem ($r = 1{\cdot}5$, $n = 1$)

θ	Damper	$k = 0{\cdot}2$	$k = 0{\cdot}5$	$k = 1{\cdot}0$	$k = 2{\cdot}0$	$k = 4{\cdot}0$
	Plane	−12·1	−31·3	−34·8	−24·8	16·2
0	Cylindrical	−3·0	−12·4	−16·9	−13·8	12·5
	Second order	−1·1	−1·5	0·3	−2·3	−4·6
	Plane	3·8	10·0	1·8	−4·8	3·6
90	Cylindrical	−0·3	−1·0	−3·7	0·2	7·
	Second order	−0·5	0·3	0·7	0·1	0·9
	Plane	28·6	13·0	−6·0	−7·7	0·6
180	Cylindrical	4·8	1·1	0·6	−2·0	−1·3
	Second order	0·5	−0·6	0·0	−0·1	−1·1

k = wave number
r = outer radius
n = the number of elements in the radial direction
θ = angle around a cylinder in degrees

Table II. Relative (percentage) errors in circular cylinder problem ($K = 1{\cdot}0$)

θ	Damper	$r=1{\cdot}5$ ($n=1$)	$r=2{\cdot}0$ ($n=2$)	$r=2{\cdot}5$ ($n=3$)	$r=3{\cdot}0$ ($n=4$)	$r=3{\cdot}5$ ($n=5$)	$r=4{\cdot}0$ ($n=6$)	$r=4{\cdot}5$ ($n=7$)	$r=5{\cdot}0$ ($n=8$)	$r=5{\cdot}5$ ($n=9$)	$r=6{\cdot}0$ ($n=10$)	$r=7{\cdot}0$ ($n=12$)
	Plane	−34·8	−9·8	2·2	6·5	5·2	1·2	−2·5	−3·1	−1·0	1·1	1·1
0	Cylindrical	−16·9	−5·1	−0·3	3·0	1·9	0·0	−1·1	−1·0	−0·2	0·6	0·1
	Second order	0·3	0·1	0·0	−0·1	0·0	0·0	0·0	0·0	0·0	0·0	0·0
	Plane	1·8	−2·9	−6·0	−5·3	−0·6	3·5	3·5	3·6	1·1	−1·8	−2·7
90	Cylindrical	−3·7	−1·4	−0·6	−0·3	−0·1	0·3	0·4	0·2	−0·1	−0·2	0·1
	Second order	0·7	0·1	−0·1	0·0	0·0	0·1	0·1	0·0	0·0	0·0	0·0
	Plane	−6·0	−8·3	−4·5	1·8	6·2	4·6	4·6	−0·8	−4·8	−3·5	−0·3
180	Cylindrical	0·6	0·7	1·0	0·6	−0·1	−0·1	−0·1	0·0	0·2	0·2	−0·2
	Second order	0·0	0·1	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0	0·0

Table III. Relative (percentage) errors in circular cylinder problem ($k = 1{\cdot}0$, $r = 7{\cdot}0$)

θ	Damper	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 12$
	Plane	5·6	2·2	1·5	1·3	1·2	1·1
0	Cylindrical	4·5	1·2	0·5	0·3	0·2	0·1
	Second order	4·3	1·0	0·4	0·2	0·1	0·0
	Plane	−8·4	−0·4	0·5	0·9	1·1	1·3
90	Cylindrical	−8·5	−1·5	−0·7	−0·4	−0·2	0·1
	Second order	−8·4	−1·5	−0·8	−0·4	−0·2	0·0
	Plane	−0·3	4·2	3·7	3·5	3·4	3·3
180	Cylindrical	−3·3	0·7	0·2	0·0	−0·1	−0·2
	Second order	−5·4	0·9	0·3	0·2	0·1	0·0

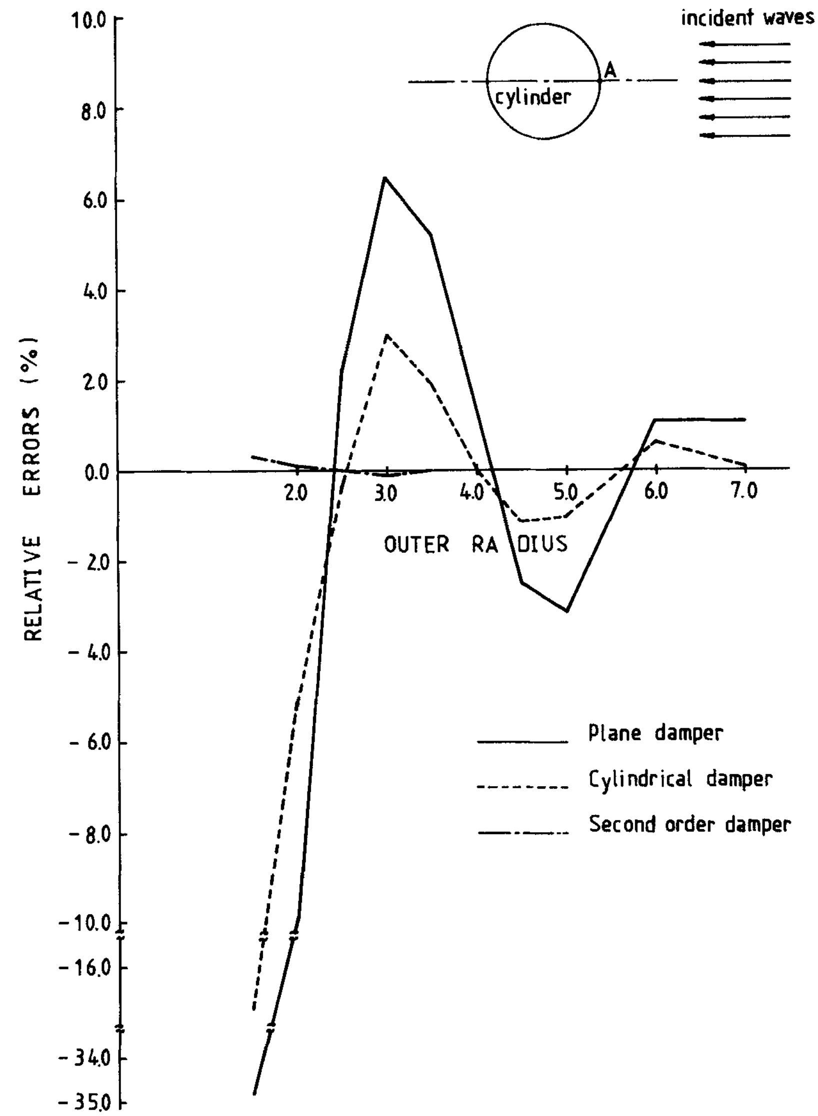

Figure 6. Comparison of relative errors for various outer radii ($k = 1{\cdot}0$). Relative error $= \frac{|\eta_n^A| - |\eta_a^A|}{|\eta_a^A|}$

results converge in an oscillatory way. The second order dampers can give very good accuracy even near the cylinder. If the allowable error is assumed to be 1 per cent, the number of elements in the radial direction is required to be $1(r_{out} = 1{\cdot}5)$ for higher order dampers, $8(r_{out} = 5{\cdot}0)$ for cylindrical dampers and more than 12 for plane dampers. Thus the outer boundary, when first order dampers (cylindrical dampers) are used, should be set up about 3 times further out than in second order dampers. It is clear that second order dampers are very effective and can reduce program size and computational cost.

The relative errors are shown in Table III when the wave number and outer radius of the FEM region are fixed and the fineness of the discretization varies. It is natural that the results should improve as the number of the elements increases. The coarse mesh, $n = 1$, which is considered as the number of elements per wavelength, always gives meaningless results and, even in the case of $n=2$, the agreement is not good and the higher order effect does not come out. On the other hand, when the fine mesh, $n=12$, is used, the agreement between theory and numerical results by cylindrical or second order dampers is extremely good. These facts indicate that even when the outer radius is far from a structure, we should divide one wavelength into 4 or more elements to obtain accurate results and when the outer radius is placed at a distance of one wavelength from a structure and the fine mesh is used, the difference between first and second order dampers vanishes.

5.2. *Elliptical cylinder problem*

It was felt that a test in which the obstacle was not itself cylindrical would show whether the improvement in results using cylindrical and higher order dampers was a real improvement or partly due to the shape of the obstacle.

Using two kinds of meshes shown in Figure 7, the elliptical problem was solved. Table IV shows the relative errors around an elliptical cylinder. There is no difference due to the angle of incident waves in any dampers' results. The numerical values by higher order dampers are extremely close to the basic ones in both cases, $n=1$ and $n=6$. In the case $n=6$, the agreement between numerical values by cylindrical dampers and basic ones is good and the errors produced using plane dampers are larger. Thus the results from the elliptical cylinder problem are quite similar to the results from the cylindrical problem.

5.3. *Sphere problem*

Waves scattered by a sphere in an infinite homogeneous domain were calculated using the meshes shown in Figure 5 ($r_{out}=1{\cdot}5$, $4{\cdot}0$, $7{\cdot}0$). The wave elevation along the surface of a sphere is plotted in Figures 8 and 9 and the relative errors are shown in Table V. In the sphere problem we have the following results which are similar to two dimensional results. Higher order dampers give quite good results and can reduce the number of finite elements, and the results by plane dampers are rather poorer.

5.4. *Ellipsoid problem*

The final problem is the same as the sphere problem except that an ellipsoid is used instead of a sphere. The numerical results are given in Table VI. As in the above problems, higher order dampers give excellent results even when the outer boundary is placed close to the structure.

6. CONCLUSION

Some infinite potential problems described by the Helmholtz equation have been solved to evaluate the accuracy of four different kinds of dampers. Of these dampers, those developed by Bayliss *et al.* are expressed in a series form and can be considered as generalizations of dampers. For example their first order dampers coincide with cylindrical dampers in two dimensional problems and spherical ones in three dimensional problems.

In this paper, four kinds of geometries were considered, i.e. circular and elliptical cylinders and spheres and ellipsoids. In all geometries the results were consistent. Cylindrical, spherical and higher order dampers are accurate and the results obtained by higher order dampers are close to theoretical values. Second order dampers typically require less than half the number of finite elements to keep the same level of accuracy as first order dampers.

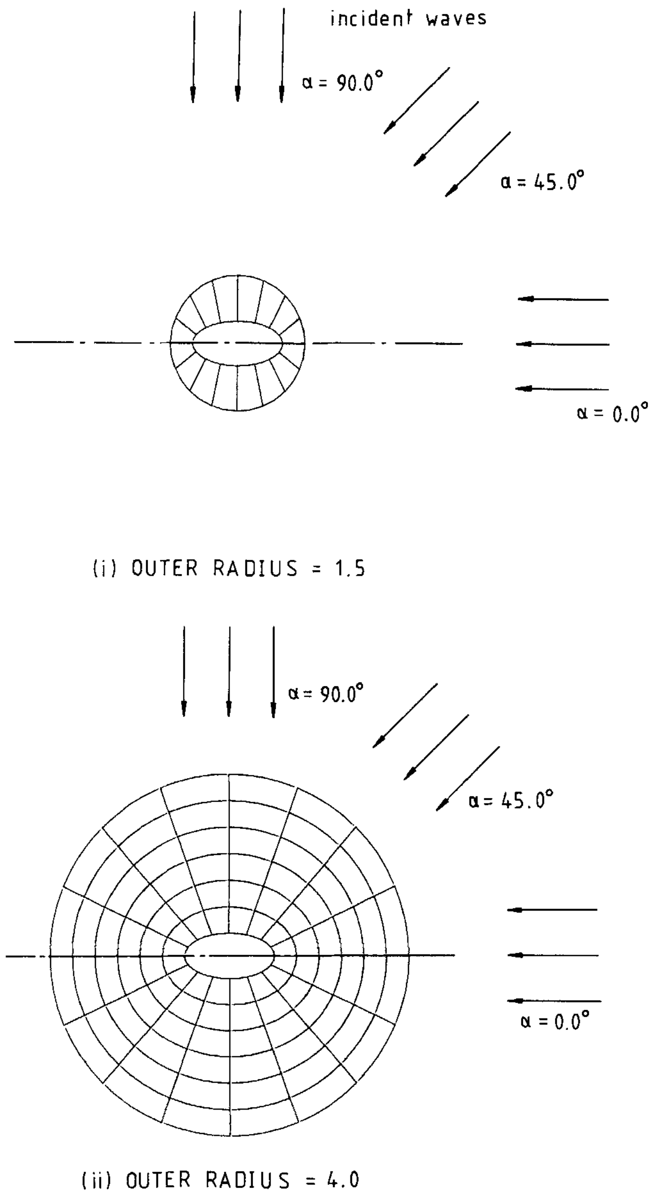

Figure 7. Element meshes

Numerical values tended to converge in an oscillatory manner as the outer radius increased. It was also found that it is usually necessary to divide one wavelength into 4 or more elements in order to obtain accurate results. Plane dampers, which are those most frequently used in practice, have the lowest accuracy.

Thus it is clear that cylindrical, spherical and higher order dampers are very effective in point of the accuracy as well as programming and computational cost, and particularly the second order damper is one of the most effective techniques for the analysis of unbounded problems. The method of Bayliss *et al.* can be strongly recommended, particularly as its extra

Table IV. Relative (percentage) errors in elliptical cylinder problem ($k = 1{\cdot}0$)

θ	Angle / Damper	$\alpha = 0{\cdot}0$ $r=1{\cdot}5$ $(n=1)$	$\alpha = 0{\cdot}0$ $r=4{\cdot}0$ $(n=6)$	$\alpha = 45{\cdot}0$ $r=1{\cdot}5$ $(n=1)$	$\alpha = 45{\cdot}0$ $r=4{\cdot}0$ $(n=6)$	$\alpha = 90{\cdot}0$ $r=1{\cdot}5$ $(n=1)$	$\alpha = 90{\cdot}0$ $r=4{\cdot}0$ $(n=6)$
	Plane	−9·1	1·2	−2·5	0·9	2·5	2·6
0	Cylindrical	−5·9	−0·2	−3·3	−0·4	−0·1	0·0
	Second order	0·4	0·1	0·4	0·0	0·1	0·0
	Plane	2·4	2·6	−3·4	−4·2	−21·5	−0·6
90	Cylindrical	−0·2	0·0	−2·0	1·8	−8·4	1·6
	Second order	0·1	0·0	0·9	0·0	0·9	0·0
	Plane	−3·8	2·7	−2·6	3·0	2·5	2·6
180	Cylindrical	0·4	−0·1	0·0	0·0	−0·1	0·0
	Second order	−0·1	0·0	−0·1	0·0	0·1	0·0
	Plane	—	—	−10·3	4·8	−13·8	4·9
270	Cylindrical	—	—	−3·6	−0·2	−4·2	−0·3
	Second order	—	—	0·1	0·0	0·2	0·0

α = angle of incident waves in degrees.

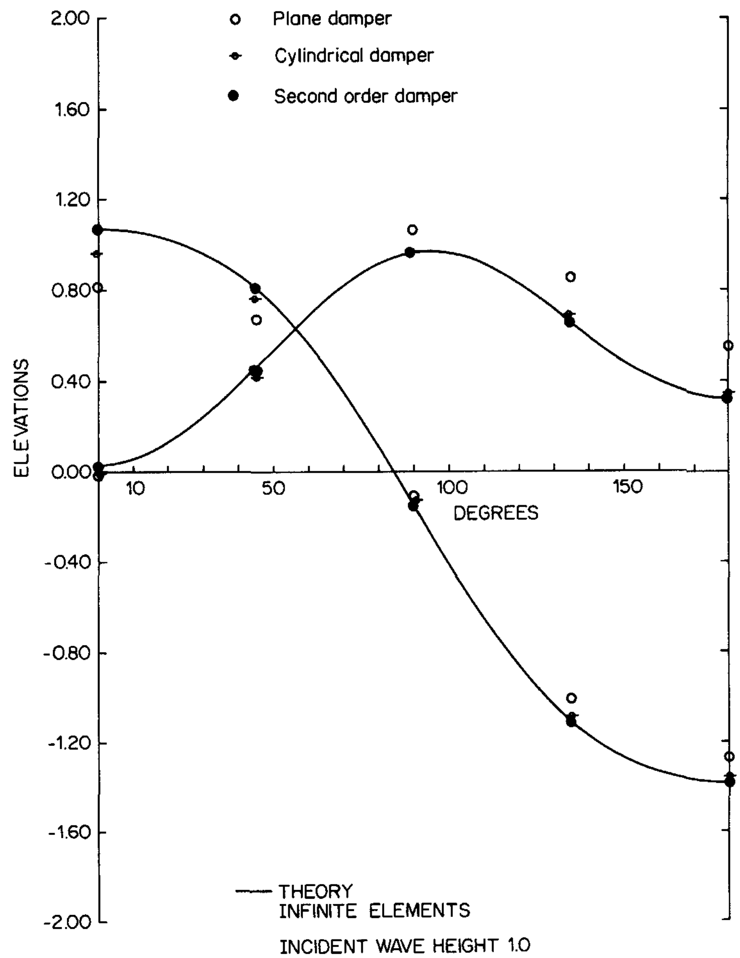

Figure 8. Comparison of wave diffraction by a sphere using three types of dampers ($k = 1{\cdot}0$, $r_{out} = 1{\cdot}5$)

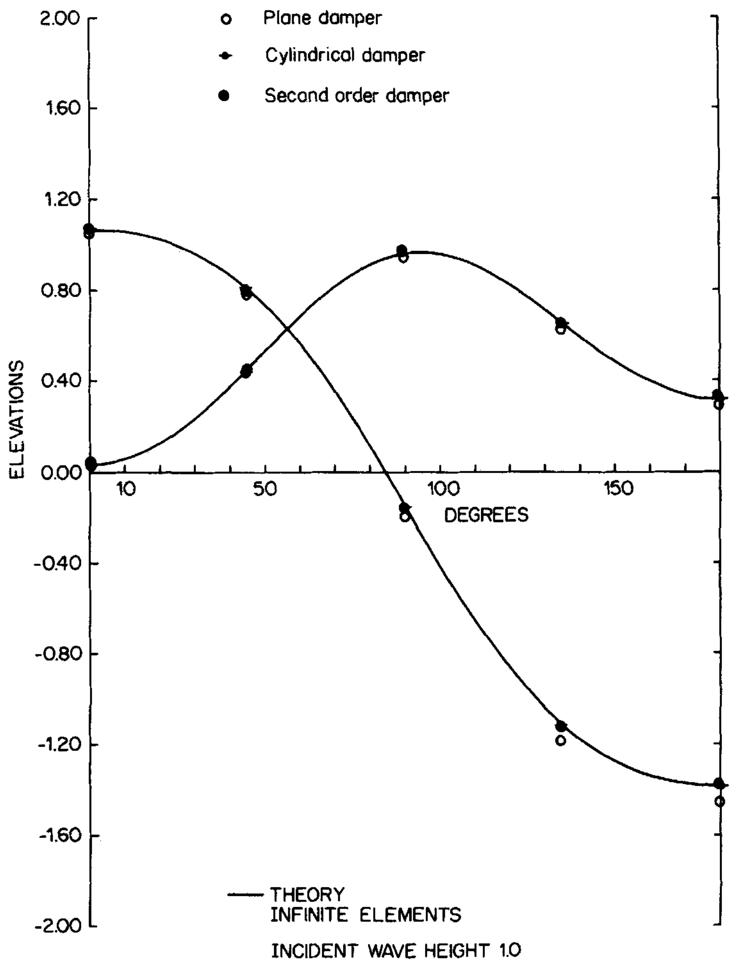

Figure 9. Comparison of wave diffraction by a sphere using three types of dampers ($k = 1{\cdot}0$, $r_{out} = 4.0$)

Table V. Relative (percentage) errors in sphere problem ($k = 1{\cdot}0$)

θ	Damper	$r = 1{\cdot}5$ ($n = 1$)	$r = 4{\cdot}0$ ($n = 6$)	$r = 7{\cdot}0$ ($n = 12$)
	Plane	−23·3	−0·7	−1·0
0	Spherical	−9·4	0·7	0·2
	Second order	0·2	−0·1	−0·1
	Plane	9·8	−0·1	−0·9
90	Spherical	0·0	0·0	0·0
	Second order	0·4	0·0	0·0
	Plane	−2·3	6·6	1·6
180	Spherical	−1·2	0·5	0·1
	Second order	−0·4	−0·1	−0·1

Table VI. Relative (percentage) errors in ellipsoid problem ($k = 1{\cdot}0$)

θ	Angle / Damper	$\alpha = 0{\cdot}0$ $r = 1{\cdot}5$ ($n = 1$)	$r = 4{\cdot}0$ ($n = 6$)	$\alpha = 90{\cdot}0$ $r = 1{\cdot}5$ ($n = 1$)	$r = 4{\cdot}0$ ($n = 6$)
	Plane	−5·2	−0·5	6·0	0·5
0	Spherical	−1·8	0·2	0·0	0·1
	Second order	0·1	0·0	0·1	0·1
	Plane	3·0	−0·4	13·5	0·6
90	Spherical	−0·1	0·0	−5·7	0·4
	Second order	−0·1	0·0	0·1	0·1
	Plane	0·6	1·4	6·0	0·5
180	Spherical	−0·8	−0·9	0·0	0·1
	Second order	−0·3	−0·1	0·1	0·1
	Plane	—	—	1·6	3·0
270	Spherical	—	—	−1·3	0·5
	Second order	—	—	−0·1	−0·1

computational expense is negligible. In view of the success of the second order dampers, it would appear to be worth while to investigate third and higher order versions, although this has obvious difficulties. It also might be feasible to develop two dimensional dampers which use the more theoretically correct Hankel functions.

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